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Discrete Mathematics 305 (2005) 379–382
DISCRETE MATHEMATICS
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Note

A note on large Cayley graphs of diameter two and given degree

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Received 25 November 2003; accepted 19 October 2005

Abstract

For a variety of infinite sets of positive integers d related to odd prime powers we describe a simple construction of Cayley graphs of diameter two and given degree d which have order close to $d^2/2$.

Keywords: Cayley graph; Degree-diameter problem

The problem of determining the largest order $n(d,k)$ of a graph of a given maximum degree d and diameter k is known as the *degree-diameter problem*. Exact values of $n(d,k)$ seem to be extraordinarily hard to obtain and are available only in rare cases. Since in this note we are interested only in graphs of diameter two, we present just a handful of most important facts related to this special case. It is well known that $n(d,2)$ is bounded above by the *Moore bound* d^2+1, with equality only if $d=2, 3, 7$, and possibly 57, see [5]. The best lower bound, based on finite fields [1], gives $n(d,2) \geqslant d^2-d+1$ if $d-1$ is a prime power (with an extra 1 added to the right-hand side if the prime is 2). For more details about the history and the current state of the degree-diameter problem we refer to the survey paper [7] and references therein.

It turns out that a number of constructions for large graphs of given degree and diameter yield vertex-transitive graphs, or even Cayley graphs. Moreover, computer search for such large graphs is almost exclusively restricted to Cayley graphs [2,4]. A challenging

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doi:10.1016/j.disc.2005.10.015

open problem is whether values close to the Moore bound can be obtained by graphs that are vertex-transitive or Cayley. This is, in a very strong sense, the case for diameter two and the degrees $d = 2$, 3, and 7, since the corresponding graphs (the pentagon, the Petersen graph, and the Hoffman–Singleton graph) are vertex-transitive and have order equal to the Moore bound $d^2 + 1$. Investigation of large vertex-transitive and Cayley graphs has thus generated an independent path of research in the degree-diameter problem.

Let $\mathrm{Cay}(d, 2)$ and $\mathrm{vt}(d, 2)$ be the largest order of a Cayley graph and a vertex-transitive graph, respectively, of degree d and diameter two. Clearly, $\mathrm{vt}(d, 2) \geqslant \mathrm{Cay}(d, 2)$ for all d. A folklore bound $\mathrm{Cay}(d, 2) \geqslant \lfloor (d+2)/2 \rfloor \lceil (d+2)/2 \rceil$, valid for all $d \geqslant 2$, results from Cayley graphs for the direct product of cyclic groups $Z_{\lfloor (d+2)/2 \rfloor} \times Z_{\lceil (d+2)/2 \rceil}$ with the generating set consisting of all pairs (x, y) in which exactly one of x, y is equal to zero [7]. In the vertex-transitive category the current record holders are the McKay–Miller–Širáň graphs [6], which are non-Cayley graphs showing that $\mathrm{vt}(d, 2) \geqslant \frac{8}{9}(d + \frac{1}{2})^2$ for all $d = (3q - 1)/2$ where q is a prime power congruent to 1 (mod 4). To the best of authors' knowledge, these were the only results on large vertex-transitive and Cayley graphs of diameter two available at the time of this writing. Other known results for Cayley graphs, such as those of [3], deal with graphs of larger diameter.

In what follows we give a surprisingly simple construction of Cayley graphs of diameter two which show that $\mathrm{Cay}(d, 2) \geqslant (d + 1)^2/2$ for an infinite set of degrees d. This improves the above folklore result by a factor of two. A modification of the construction yields an infinite number of other infinite sets of degrees and leads to a bound which is, in some sense, arbitrarily close to $d^2/2$. We point out that all these degree sets are different from the one associated with the previously mentioned result of [6].

It should be no surprise that our construction is based on finite fields. Let $F = GF(q)$ be the Galois field of an odd prime power order q. We denote by F^+ the additive group of F. The direct product $F^+ \times F^+$ has an automorphism α of order two that interchanges each element with its negative. We will be working with the extension of the group $F^+ \times F^+$ by α, that is, with the semidirect product $G = (F^+ \times F^+) \rtimes \langle \alpha \rangle$. Elements of the group G will be written in the form (a, b, i) where $i \in \{1, -1\} \cong Z_2$. Multiplication in G is then given by $(a, b, i)(a', b', i') = (a + ia', b + ib', ii')$. Observe that the elements $(a, b, -1)$ are involutions for all $a, b \in F^+$.

Let $X = \{(a, a^2, -1);\ a \in F^+\} \cup \{(0, b, 1);\ b \in F^+,\ b \neq 0\}$. Clearly, the set X is a generating set for G and is closed under inverses, that is, $X = X^{-1}$. We therefore may form the (undirected) Cayley graph $\Gamma = \mathrm{Cay}(G, X)$. We will occasionally write $\Gamma(q)$ instead of Γ to emphasize the dependence of the Cayley graph on q.

Theorem 1. *The graph $\Gamma(q)$ has diameter two for any odd prime power q.*

Proof. Since $\Gamma(q)$ is a Cayley graph, it is sufficient to show that any nonidentity element of $G \backslash X$ is a product of two generators from X. This is clear for elements of the form $(a, b, -1) \in G$ where $b \neq a^2$ because $(a, b, -1) = (0, b - a^2, 1)(a, a^2, -1)$. It remains to consider triples $(a, b, 1)$ with $a \neq 0$. Let $g, h \in F^+$ be the unique solution of the linear system $g - h = a$, $g + h = a^{-1}b$. Then $(g, g^2, -1)(h, h^2, -1) = (g - h, g^2 - h^2, 1) = (a, b, 1)$, which shows that such triples are also products of two generators from X. $\square$

The graph $\Gamma = \Gamma(q)$ has order $|G| = 2q^2$ and degree $d = |X| = 2q - 1$. Therefore, $\mathrm{Cay}(d, 2) \geqslant (d + 1)^2/2$ for any d such that $d = 2q - 1$ where q is an odd prime power.

In order to obtain bounds close to $d^2/2$ for other infinite sequences of degrees one may simply add involutions to the generating set X of the group G. For example, adding exactly one involution to X yields a Cayley graph of the same order $2q^2$ and diameter two, but of degree $2q$. It follows that $\mathrm{Cay}(d, 2) \geqslant d^2/2$ for any even d such that $d/2$ is an odd prime power.

The disadvantage of adding more involutions is the increasing degree while the order of the graphs remains the same. Nevertheless, this process still yields bounds of the form $\mathrm{Cay}(d, 2) \geqslant \left(\frac{1}{2} - \varepsilon\right) d^2$ for infinitely many distinct infinite sequences of degrees.

Theorem 2. *For any integer $t \geqslant 2$ and any arbitrarily small $\varepsilon > 0$ there exists an integer m such that for all odd prime powers $q \geqslant m$ and all d of the form $d = 2q - 1 + t$ we have* $\mathrm{Cay}(d, 2) \geqslant \left(\frac{1}{2} - \varepsilon\right) d^2$.

Proof. For any ε such that $0 < \varepsilon < \frac{1}{2}$ and for any integer $t \geqslant 2$ let $m = \lceil (t - 1)c/(2 - 2c) \rceil$ where $c = \sqrt{1 - 2\varepsilon}$. Observe that if ε is sufficiently small, then $m > t$. Take any odd prime power q such that $q \geqslant m$. Using the previously introduced notation, let $Y \subset G$ be an arbitrary subset of the set $\{(1, b, -1);\ b \in F^+,\ b \neq 1\}$ such that $|Y| = t$; note that $Y \cap X = \emptyset$. As indicated above, we will consider the Cayley graph $\Gamma_Y = \mathrm{Cay}(G, X \cup Y)$, which is well defined since Y consists just of involutions. The graph Γ_Y is a supergraph of Γ (on the same vertex set). By Theorem 1, Γ_Y has diameter two; its degree is $d = |X \cup Y| = 2q - 1 + t$. One can check that the inequality $q \geqslant m$ implies $(d + 1 - t)^2 \geqslant (1 - 2\varepsilon)d^2$, or, equivalently, $2q^2 \geqslant (1/2 - \varepsilon)d^2$. Since $2q^2$ is the order of the graph Γ_Y, our theorem follows. □

We remark that the Cayley graphs of Theorem 1 can be obtained as lifts of dipoles (with voltages in $F^+ \times F^+$) in a way similar to the construction of the McKay–Miller–Širáň graphs given in [8]. In this connection it is interesting to mention the result of [9] that sets an upper bound for this type of construction: If a dipole with a voltage assignment in an abelian group lifts to a graph of large degree d and diameter two, then its order never exceeds $0.932d^2$.

Acknowledgements

Research of the second author was sponsored by the US Slovak Science and Technology Joint Fund under Project number 020/2001. Partial support by the VEGA Grant 1/9176/02 and the APVT Grant 20-023302 is acknowledged by both authors. The first author also thanks NZIMA and the Departments of Mathematics at University of Auckland and at Syracuse University for hospitality during preparation and completion of this work.

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